# The Centroidal Voronoi Tessellation (CVT): an Application to Optimal Placement of Resources

Annie Kendzior
Advisor: Dr. Hoa Nguyen
Mathematics Department
Trinity University

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# Outline

# Motivation

We are seeking the optimal placement of nine high schools in a county so as to make them the most convenient for students. We make the following assumptions:

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- The transportation cost to the students, as a whole, is measured by the distance to the nearest school averaged over all students in the region.
- The optimal placement of schools is defined to be the one that minimizes the total transportation cost to the students.

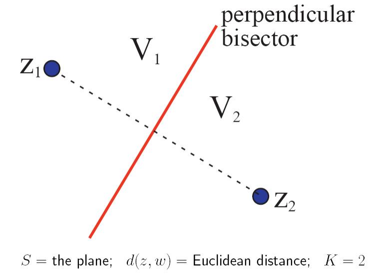
# Motivation

This problem is an application of the centroidal Voronoi tessellation (CVT) concept. It turns on that the optimal placement of the schools is at the centroids of a CVT of the city with respect to a given density function of the student population.

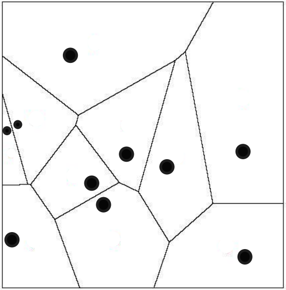
# Voronoi Region

Given a set of generators $\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_K$ belonging to a set $\mathbf{S}$, the Voronoi region, $V_j$, corresponding to the generator $\mathbf{z}_j$ is defined by:

$$V_j = \{\mathbf{w} \in \mathbf{S} : d(\mathbf{w}, \mathbf{z}_j) < d(\mathbf{w}, \mathbf{z}_i), i = 1, ..., K, i \neq j\}$$




A Voronoi tessellation of 2 generators

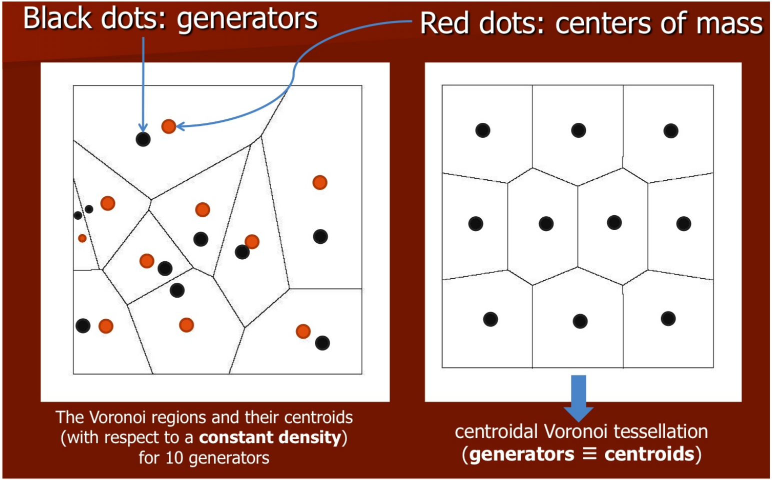


# Voronoi Tessellation (VT)

Given an open set $\Omega \subseteq \mathbf{R}^n$, the set $\{V_i\}_{i=1}^{K}$ is called a tessellation of $\Omega$ if $V_i \cap V_j = 0$ for $i \neq j$ and $\bigcup_{i=1}^{K} \overline{V_i} = \overline{\Omega}$.

The set $\{V_i\}_{i=1}^{K}$ is called a Voronoi tessellation or Voronoi diagram of $\Omega$.

# VT vs. CVT



Black dots: generators
Red dots: centers of mass
The Voronoi regions and their centroids
(with respect to a **constant density**)
for 10 generators
centroidal Voronoi tessellation
(**generators ≡ centroids**)

# Center of Mass/Centroid

Given a non-negative and almost everywhere continuous density function $\rho(\mathbf{x})$ defined on $\Omega$ and given any region $V \subset \Omega$, we define its centroid or center of mass by

$$\bar{\mathbf{z}} = \frac{\int_V \mathbf{x}\rho(\mathbf{x})d\mathbf{x}}{\int_V \rho(\mathbf{x})d\mathbf{x}}$$

In particular, for each Voronoi region $V_i$, $i = 1, ..., K$ we can define its centroid $\overline{\mathbf{z}_i}$ by

$$\overline{\mathbf{z}_i} = \frac{\int_{V_i} \mathbf{x}\rho(\mathbf{x})d\mathbf{x}}{\int_{V_i} \rho(\mathbf{x})d\mathbf{x}}$$

# Optimization Problem

CVT = solution of an optimization problem.

**Energy/Cost Function** Given $K$ generators $\{\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_K\}$ and associated $V = \{V_1, V_2, ..., V_K\}$, and a density function $\rho(\mathbf{x})$ on $\Omega \subseteq \mathbf{R}^n$, we define the energy/cost function:

$F(\{\mathbf{z}_k, V_k\}_{k=1}^K) = \sum_{k=1}^K \int_{V_k} \rho(\mathbf{x})|\mathbf{x} - \mathbf{z}_k|^2 d\mathbf{x}$

# Proposition

Given an integer $K > 1$ and a non-negative and almost everywhere continuous density function $\rho(\mathbf{x})$ defined on $\Omega \subseteq \mathbf{R}^n$. Let $\{V_i\}_{i=1}^{K}$ denote an arbitrary subdivision of $\Omega$ into $K$ non-overlapping, covering subsets and let $\{z_i\}_{i=1}^{K}$ denote an arbitrary set of $K$ points in $\overline{\Omega}$.

**Then, a necessary condition for $F(\{\mathbf{z}_i, V_i\}_{i=1}^{K})$ to be minimized is that $\{\mathbf{z}_i, V_i\}_{i=1}^{K}$ define a centroidal Voronoi tessellation of $\Omega$.**

We see that $F(\cdot)$ is a variance measure; we will refer to it as the CVT energy.

# Proof

Given $F(\{\mathbf{z}_k, V_k\}_{k=1}^{K}) = \sum_{k=1}^{K} \int_{V_k} \rho(\mathbf{x})|\mathbf{x} - \mathbf{z}_k|^2 d\mathbf{x}$,

$$\begin{aligned}\frac{\partial F}{\partial z_{im}} &= \int_{V_i} \rho(\mathbf{x}) \frac{\partial}{\partial z_{im}} |\mathbf{x} - \mathbf{z}_i|^2 d\mathbf{x} \\ &= \int_{V_i} \rho(\mathbf{x}) 2(x_m - z_{im})(-1) d\mathbf{x} \\ &= -2 \int_{V_i} \rho(\mathbf{x})(x_m - z_{im}) d\mathbf{x}\end{aligned}$$

where $i = 1, ..., K$ and $m = 1, ..., n$ since $\Omega \subseteq \mathbf{R}^n$.

## Proof

Set $\frac{\partial F}{\partial z_{im}} = 0$ i.e.,

$$
\begin{aligned}
-2\int_{V_i} \rho(\mathbf{x})(x_m - z_{im})d\mathbf{x} = 0 \\
\int_{V_i} \rho(\mathbf{x})x_m d\mathbf{x} = \int_{V_i} \rho(\mathbf{x})z_{im} d\mathbf{x} \\
z_{im} = \frac{\int_{V_i} \rho(\mathbf{x})x_m d\mathbf{x}}{\int_{V_i} \rho(\mathbf{x})d\mathbf{x}} = z_{im}^*
\end{aligned}
$$

which is the $m$-th coordinate of the centroid of $V_i$.

# Numerical Simulation: Lloyd's Algorithm

Given a set $\Omega$, a positive integer $K$, and a density function $\rho$ defined on $\Omega$,

1. select an initial set of $K$ points $\{\mathbf{z}_i\}_{i=1}^{K}$;

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3. compute the mass centroids of the Voronoi regions $\{V_i\}_{i=1}^K$ found in Step (2); these centroids are the new set of points $\{\mathbf{z}_i\}_{i=1}^K$.

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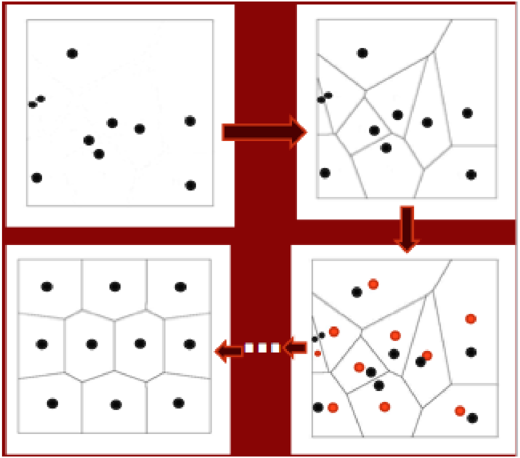
4. If this new set of points meets some convergence criterion, terminate; otherwise, return to Step (2).

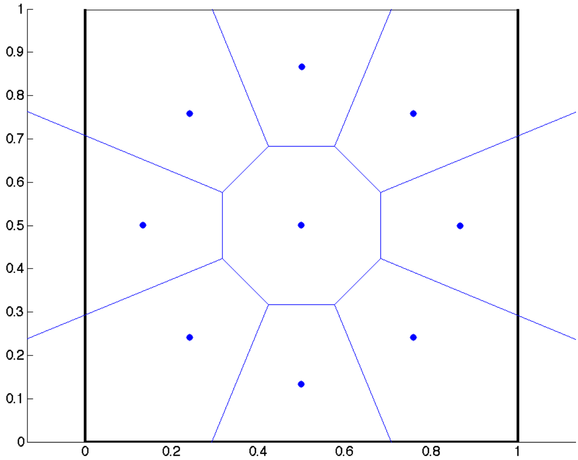
# Lloyd's Algorithm Steps



# Numerical Results

In our problem, we focused on nine individual high schools that were to be placed in a given county. The county has an area shaped like a unit square. The high schoolers in the county are evenly distributed throughout the area, giving this problem a uniform density.

# CVT Result with Uniform Density



# Numerical Results

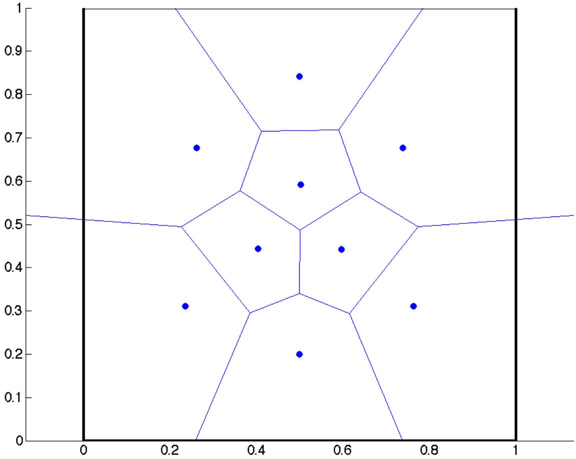
Now, a more realistic example that is similar to ours would be that the high schoolers are not evenly distributed, therefore there is no uniform density.

For example, students tend to live closer to the center of the county which is the center of the square. As a result, we want the schools to be closer to the center which means we must use the density function that has a higher value near the center and a smaller value near the boundary of the county.

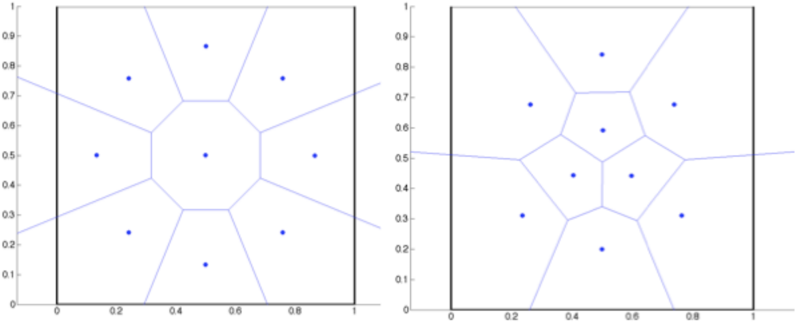
$$\rho(\mathbf{x}) = e^{10d(\mathbf{x},\partial\Omega)}$$

where $d(\mathbf{x}, \partial\Omega)$ is the distance from the point, $\mathbf{x}$, to the closest boundary of the domain, $\Omega$.

# CVT Result with Non-Uniform Density



# Comparison




Left: uniform CVT; Right: non-uniform CVT

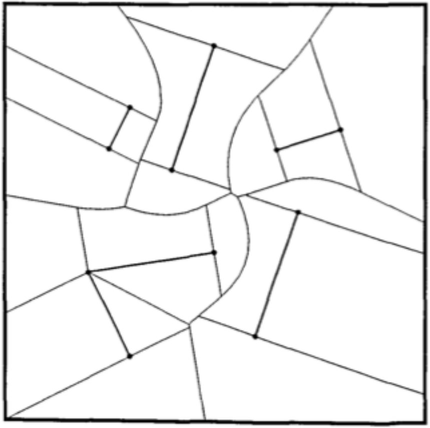
# Conclusion

From this optimal location of high schools we learned that CVT can be used for different optimization problems. Depending on a specified density function, CVT results can reflect the underlying distribution over a given domain.

We can also generalize CVT to solve the locational-optimization problems of line-like and area-like generators (Okabe).

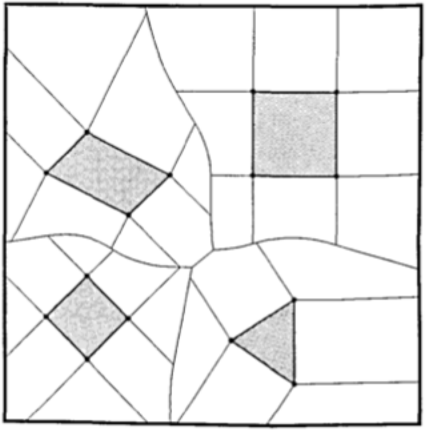
# Line-like Generators

Use line generators instead of point generators to account for the size of the school. See the following figure (Okabe).



# Area-like Generators

To take the architecture into account (i.e.. shape of campus), rather than using point generators, area generators can be used. See the following figure (Okabe).



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# References


- H. Nguyen, J. Burkardt, M. Gunzburger, L. Ju, and Y. Saka. "Constrained CVT meshes and a comparison of triangular mesh generators," Computational Geometry: Theory and Applications, Vol. 42, pp. 1-19, 2009.
- Q. Du, V. Faber, and M. Gunzburger. "Centroidal Voronoi tessellations: Applications and algorithms," SIAM Rev. 41, pp 637-676, 1999.
- A. Okabe. "Locational Optimization Problems Solved Through Voronoi Diagrams," European Journal of Operational Research, Vol. 98, pp. 445-456, 1997.